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The Anosov theorem for flat generalized Hantzsche–Wendt manifolds

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Abstract

In this paper we prove that $N(f) = |L(f)|$ for any continuous map f on a given orientable flat generalized Hantzsche–Wendt manifold. This is the analogue of a theorem of Anosov for continuous maps on nilmanifolds. We also show that the theorem always fails in the non-orientable case.

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1. Introduction

Let M be a smooth closed manifold and let $f : M \to M$ be a continuous self-map of M. In fixed point theory, two numbers are associated with f to provide information on its fixed points: the Lefschetz number $L(f)$ and the Nielsen number $N(f)$. Inspired by the fact that $N(f)$ gives more information than $L(f)$, but unfortunately $N(f)$ is not readily computable from its definition (while $L(f)$ is much easier to calculate), in literature, a considerable amount of work has been done on investigating the relation between both numbers. Anosov [1] proved that $N(f) = |L(f)|$ for all continuous maps $f : M \to M$ if M is a nilmanifold, but he also observed that there exists a continuous map $f : K \to K$ of the Klein bottle K such that $N(f) \neq |L(f)|$.

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There are two possible ways of trying to generalize this theorem of Anosov. Firstly, one can search classes of maps for which the relation holds for a specific type of manifold. For instance, Kwasik and Lee proved [10] that the Anosov theorem holds for homotopic periodic maps of infra-nilmanifolds and Malfait [12] did the same for virtually unipotent maps of infra-nilmanifolds. Secondly, one can look for classes of manifolds, other than nilmanifolds, for which the relation holds for all continuous maps of the given manifold, as was established by Keppelmann and McCord for exponential solvmanifolds [8].

In this article we follow the second approach and we show that $N(f) = |L(f)|$ holds for all continuous maps f of an orientable flat generalized Hantzsche–Wendt manifold. In this way we obtain the first known example of a large class of flat manifolds, outside the class of the tori, for which the theorem of Anosov always holds. To see that this is really a large class, we refer to the work of Miatello and Rossetti [13], where it is shown that the number of orientable flat generalized Hantzsche–Wendt manifolds grows exponentially with the dimension (while there is only one torus in each dimension!).

We also show that it is essential that M is orientable, since for any non-orientable flat generalized Hantzsche–Wendt manifold M we construct a continuous map $f : M \to M$ such that $N(f) \neq |L(f)|$.

2. Preliminaries

An affine endomorphism of $\mathbb{R}^n$ is an element (a, A) of $\mathbb{R}^n \rtimes M_n(\mathbb{R})$ with $a \in \mathbb{R}^n$ the translational part and $A \in M_n(\mathbb{R})$ (= the semigroup of $n \times n$ matrices) the linear part. The product of two affine endomorphisms is given by $(a, A)(b, B) = (a + Ab, AB)$ and (a, A) maps an element $x \in \mathbb{R}^n$ to $a + Ax$. If the linear part A belongs to $\mathrm{Gl}(n, \mathbb{R})$, then (a, A) is an affine transformation of $\mathbb{R}^n$. We write $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{Gl}(n, \mathbb{R})$ for the group of affine transformations of $\mathbb{R}^n$.

2.1. Flat manifolds and continuous maps

Let M be a flat (Riemannian) manifold of dimension n and assume $E = \pi_1(M)$ denotes its fundamental group. Then E is a torsion-free group fitting into an extension $0 \to \mathbb{Z}^n \to E \to F \to 1$, where $\mathbb{Z}^n$ is maximal abelian in E and F is a finite group. Equivalently, E is a uniform, discrete subgroup of $\mathbb{R}^n \rtimes \mathrm{O}(n) \subseteq \mathrm{Aff}(\mathbb{R}^n)$, acting freely on $\mathbb{R}^n$. The group $\Lambda = E \cap \mathbb{R}^n$ of pure translations in E is a uniform lattice of $\mathbb{R}^n$ and so $E \cap \mathbb{R}^n \cong \mathbb{Z}^n$ (the free abelian group which is the kernel of the short exact sequence above). We refer to F as the holonomy group of $M = E \backslash \mathbb{R}^n$ and E is said to be a Bieberbach group. The holonomy F acts on $\mathbb{Z}^n$ by conjugation in E, defining a faithful representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$, which is referred to as the holonomy representation (and which is defined up to conjugation inside $\mathrm{Gl}(n, \mathbb{Z})$, depending on the choice of the free generating set of $E \cap \mathbb{R}^n$). The faithfulness of the holonomy representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ is equivalent to $\mathbb{Z}^n$ being maximal abelian in E. A standard reference is [4].

Essential for our purposes is the following result due to Lee [11] (formulated here only for Bieberbach groups, not for the more general case of almost-crystallographic groups).

Theorem 2.1. *Let $E, E' \subset \mathrm{Aff}\mathbb{R}^n$ be two Bieberbach groups. Then for any homomorphism $\theta : E \to E'$, there exists a $g = (d, D) \in \mathbb{R}^n \rtimes M_n(\mathbb{R})$ such that $\theta(\alpha) \cdot g = g \cdot \alpha$ for all $\alpha \in E$.*

Important for us, is the following corollary of this theorem (we refer to [6] for a detailed proof).

Corollary 2.2. *Let $M = E\backslash\mathbb{R}^n$ be a flat Riemannian manifold and $f : M \to M$ is a continuous map of M. Then f is homotopic to a map $h : M \to M$ induced from an affine endomorphism $g : \mathbb{R}^n \to \mathbb{R}^n$.*

We say that g is the homotopy lift of f. Note that one can find the homotopy lift of a given f, by using Theorem 2.1 for the homomorphism $f_* : \pi_1(M) \to \pi_1(M)$ induced by f. In this way, we obtain a method to characterize all continuous maps, up to homotopy, of a flat Riemannian manifold M by constructing all possible homomorphisms θ of $\pi_1(M)$ and so all suitable affine endomorphisms of $\mathbb{R}^n$, namely the affine endomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ referred to in Corollary 2.2.

2.2. The Nielsen numbers

Let M be a compact manifold and assume $f : M \to M$ is a continuous map. The Lefschetz number $L(f)$ is defined by

$$L(f) = \sum_i (-1)^i \mathrm{Trace}(f_* : H_i(M, \mathbb{Q}) \to H_i(M, \mathbb{Q})).$$

The set $\mathrm{Fix}(f)$ of fixed points of f is partitioned into equivalence classes, referred to as fixed point classes, by the relation: $x, y \in \mathrm{Fix}(f)$ are f-equivalent if and only if there is a path w from x to y such that w and fw are homotopic. To each class one assigns an integer index. A fixed point class is said to be essential if its index is non-zero. The Nielsen number of f is the number of essential fixed point classes of f. The relation between $L(f)$ and $N(f)$ is given by the property that $L(f)$ is exactly the sum of the indices of all fixed point classes. For more details we refer to [3,7,9].

In this article we examine the relation $N(f) = |L(f)|$ for continuous maps $f : M \to M$. Since $L(f)$ and $N(f)$ are homotopy invariants, we know that calculating $L(f)$ and $N(f)$ for continuous maps f on a flat manifold M, is equivalent to computing the Nielsen numbers for the continuous maps of M which are induced by affine endomorphisms (Corollary 2.2). We will do this using the following theorem of Lee [11] (in the special case of flat Riemannian manifolds).

Theorem 2.3. *Let $f : M \to M$ be a continuous map on a flat manifold M and let $T : F \to \mathrm{Gl}(n, \mathbb{Z})$ be the associated holonomy representation. Let $g = (d, D) \in \mathbb{R}^n \rtimes M_n(\mathbb{R})$ be a homotopy lift of f. Then $N(f) = L(f)$, resp. $N(f) = -L(f)$, if and only if $\det(I_n - T(x)D) \geq 0$, resp. $\det(I_n - T(x)D) \leq 0$, for all $x \in F$.*

2.3. Generalized Hantzsche–Wendt manifolds

We start with the following definitions.

Definition 2.4. An n-dimensional flat manifold is called a Generalized Hantzsche–Wendt (GHW) manifold if its holonomy group is isomorphic to $\mathbb{Z}_2^{n-1}$. In this case the Bieberbach group $E = \pi_1(M)$ is called a GHW group.

Definition 2.5. An n-dimensional Bieberbach group E is said to be diagonal if its lattice Λ of translations has an orthogonal basis for which the holonomy representation T diagonalizes.

Rossetti and Szczepański [15] proved the following theorem.

Theorem 2.6. *The fundamental group $\pi_1(M)$ of a generalized Hantzsche–Wendt manifold is diagonal*.

Remark 2.7. This paper depends strongly on the above theorem. The reader who is uncomfortable with the fact that we have to refer to a not yet published paper for this result, can include the fact that the holonomy representation is diagonalizable as part of the definition of a GHW manifold or (s)he can add this as an extra condition to all the results which follow.

Before analyzing the consequences of Theorem 2.6, let us recall the following remark.

Remark 2.8. Suppose M is an n-dimensional flat manifold with associated holonomy representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$. Then M is an orientable manifold if and only if $\det(T(x)) = 1$ for all $x \in F$. In case there exists an element $x \in F$ such that $\det(T(x)) = -1$, M is called a non-orientable manifold. For more background, see [2, p. 211; 5, p. 135].

Suppose M is an n-dimensional flat GHW manifold and $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ is the associated holonomy representation. Because of Theorem 2.6 we may assume that $T(x)$ is diagonal for each $x \in \mathbb{Z}_2^{n-1}$ and hence we know that the diagonal elements must be 1 or -1.

If moreover, M is an orientable manifold, then for each $x \in \mathbb{Z}_2^{n-1}$, the diagonal entries of $T(x)$ consist of an even number of -1's while the others are 1. In fact it is obvious that in $\mathrm{Gl}(n, \mathbb{Z})$ there are exactly 2^{n-1} diagonal matrices whose diagonal entries consist of an even number of -1's while the other entries are 1. We obtain the following corollary.

Corollary 2.9. *Let M be an n-dimensional orientable flat GHW manifold and $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ its associated holonomy representation. Then*

1. *The image of $T : \mathbb{Z}_2^{n-1} \to \mathrm{Gl}(n, \mathbb{Z})$ is completely determined*;
2. *n is an odd integer*;
3. *the first Betti number of M is 0*.

Proof. Note that, because of Theorem 2.6, T is a diagonal representation. Since a holonomy representation must be faithful and the holonomy group is of order 2^{n-1}, the first result follows immediately.

Suppose n is an even integer, then there exist a $x \in \mathbb{Z}_2^{n-1}$ such that $T(x) = -Id$ which is not possible since $\pi_1(M)$ is torsion-free.

One can easily verify (e.g. using [5, p. 143]) that the first Betti number of an orientable flat GHW manifold must be 0. □

In an analogous way, one can consider the class of non-orientable flat GHW manifolds M. For these manifolds, it is easy to prove that the first Betti number of M must be 0 or 1. In the latter case again the image of the holonomy representation is completely determined. In the former case there are more possibilities. In [15] the authors show that there are $n/2$ possibilities for n even and $(n+1)/2$ possibilities for n odd. For more details we refer to [13,15].

To finish the preliminaries, we already note the following for flat GHW manifolds.

Theorem 2.10. *Let M be a flat GHW manifold and let $f : M \to M$ be a self-homotopy equivalence of M, then $N(f) = L(f)$.*

Proof. Rossetti and Szczepański [15] proved that the outer automorphism group $\mathrm{Out}(\pi_1(M))$ of the fundamental group of a flat GHW manifold is finite. Therefore it follows from [12] that each self-homotopy equivalence is homotopically periodic. And because of [10], we obtain that $N(f) = L(f)$. □

3. Orientable flat GHW manifolds

The goal of this section is to prove the Anosov theorem for orientable flat GHW manifolds M, i.e. to show that the relation $N(f) = |L(f)|$ holds for any continuous map $f : M \to M$ (Theorem 3.7).

Let M be an orientable n-dimensional flat GHW manifold with fundamental group $E = \pi_1(M)$ and associated holonomy representation $T : F \to \mathrm{Gl}(n, \mathbb{Z})$. $T(F)$ is exactly the set of the 2^{n-1} diagonal matrices with 1 or -1 on the diagonal and such that the number of -1's is even (and the number of 1's is odd, as the dimension of M must be odd, Corollary 2.9).

The group $T(F)$ is hence generated by the set of diagonal matrices A_i $(1 \le i \le n-1)$, where all the diagonal entries are -1, except the entry on the ith row and column, which is 1. As a consequence, we can assume that the group E is generated by

$$(z_1, I_n), \ldots, (z_n, I_n), (a_1, A_1), \ldots, (a_{n-1}, A_{n-1}), \tag{1}$$

where $z_i \in \mathbb{Z}^n$ $(1 \le i \le n)$ and $a_i \in \mathbb{R}^n$ are appropriate translational parts. Let us, for the rest of this paper, denote the kth component of an element $b \in \mathbb{R}^n$ by b^k.

Miatello and Rossetti showed in [14, Lemma 1.4] that we can assume that the elements a_i^k of a_i are 0 or 1/2. Although this result does not allow us to specify the translational parts a_i completely, we do already know that a_i^i must be 1/2 for each i. Indeed, if $a_i^i = 0$, a simple computation shows that

$$(a_i, A_i) \cdot (a_i, A_i) = (a_i + A_i a_i, A_i^2) = \left(\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}, I_n \right),$$

which would imply that E has torsion.

We will refer to a generating set (1) of E, with the $a_i^k = 0$ or $1/2$ as a suitable generating set.

Remark 3.1. Let $A_n = A_1 \cdot A_2 \cdots A_{n-1}$. Then A_n is also a diagonal matrix with all diagonal entries equal to -1, except the last one which is 1 (since n is odd). Again because of [14, Lemma 1.4] we can assume that there exists a $a_n \in \mathbb{R}^n$ with components 0 or $1/2$ such that $(a_n, A_n) \in E$.

As already mentioned above, in order to prove the Anosov theorem for flat orientable GHW manifolds, it is sufficient to deal with the continuous maps of M which are induced by a suitable affine endomorphism of $\mathbb{R}^n$. Therefore we need a full description of all affine endomorphisms $g : \mathbb{R}^n \to \mathbb{R}^n$ which are obtained in Corollary 2.2.

Lemma 3.2. *Let M be an orientable n-dimensional flat GHW manifold* $(n \geq 3)$ *and let* $(z_1, I_1), \ldots, (z_n, I_n), (a_1, A_1), \ldots, (a_{n-1}, A_{n-1})$ *be a suitable generating set of* $\pi_1(M) = E$. *Assume θ is a homomorphism of E and* $(d, D) \in \mathbb{R}^n \rtimes M_n(\mathbb{R}^n)$ *is a suitable affine endomorphism* (i.e. $\forall \alpha \in E : \theta(\alpha) \cdot (d, D) = (d, D) \cdot \alpha$). *Denoting the* (i, j)*th entry of D by d_{ij} we then have the following*:

1. *If there exists a* $j \in \{1, 2, \ldots, n\}$ *such that* $\theta(a_j, A_j) = (z, I_n)$, *with* $z \in \mathbb{Z}^n$ (*the image of some* (a_j, A_j) *is a pure translation*), *then for all* $i\,(1 \leq i \leq n)$, $d_{ik} = 0$ *if* $k \neq j\,(1 \leq k \leq n)$, *while d_{ij} is an even integer*.
2. *If there exists a* $j \in \{1, 2, \ldots, n\}$ *such that* $\theta(a_j, A_j) = (z, I_n)(b, B)$, *with* $z \in \mathbb{Z}^n$, $b \in \mathbb{R}^n$ *a translation consisting of 0's and $1/2$'s and $B \neq I_n$ a finite product of A_i's* (*the image of some* (a_j, A_j) *is not a pure translation*), *then there exists a* $i\,(1 \leq i \leq n)$ *such that d_{ij} is an odd integer and* $d_{ik} = 0$ *for all* $k \neq j\,(1 \leq k \leq n)$.

Proof.

1. Since $\theta(a_j, A_j) \cdot (d, D) = (z, I_n) \cdot (d, D) = (z + d, D)$ and $(d, D) \cdot (a_j, A_j) = (d + Da_j, DA_j)$, it follows that $D = DA_j$ and $z + d = d + Da_j$. Now, A_j only has $+1$ in the jth column while the other diagonal entries are -1, this forces all the columns of D to be zero, except the jth column.

   For the translational parts we must have that $z + d = d + Da_j$. Since $a_j^j = 1/2$ and only the jth column of D is non-zero, it follows that $(z^1 + d^1 \cdots z^n + d^n)^t = (d^1 + (1/2)d_{1j} \cdots d^n + (1/2)d_{nj})^t$. So d_{ij} must be an even integer for all $i\,(1 \leq i \leq n)$.

2. B is a finite product of A_i's, so B is also a diagonal matrix with 1's and -1's as diagonal entries. As above we denote the (i, j)th entry of B by b_{ij}. Note that $(b, B) \cdot (b, B) = (b + Bb, I_n)$, from which it follows that $b + Bb$ cannot be equal to zero. This implies that there exists a i $(1 \leq i \leq n)$ such that $b_{ii} = 1$ and $b^i = 1/2$.

   Since $\theta(a_j, A_j) \cdot (d, D) = (z + b, B) \cdot (d, D) = (z + b + Bd, BD)$ and $(d, D) \cdot (a_j, A_j) = (d + Da_j, DA_j)$, we have that $BD = DA_j$ and $z + b + Bd = d + Da_j$.

   Since $b_{ii} = 1$, the ith row of BD equals the ith row of D. Similarly the jth column of DA_j equals the jth column of D while the other columns of DA_j are equal to minus the corresponding column of D. It follows that $d_{ik} = 0$ for all $k \neq j$ $(1 \leq k \leq n)$.

   To show that d_{ij} is an odd integer, we look at the translational parts for which we know that $z + b + Bd = d + Da_j$. Again, using $b_{ii} = 1$, the ith component of the above equality reduces to

$$z^i + b^i + d^i = d^i + \sum_{k=1}^{n} d_{ik} a_j^k = d^i + d_{ij} a_j^j = d^i + \frac{1}{2} d_{ij}.$$

   Since $b^i = 1/2$ this shows that d_{ij} must be an odd integer. □

Using the lemma above, we can now prove the following proposition in which we describe the linear parts of all suitable affine endomorphisms of $\mathbb{R}^n$.

Proposition 3.3. *Let M be an orientable n-dimensional flat GHW manifold with fundamental group $\pi_1(M) = E$ $(n \geq 3)$. Let θ be a homomorphism of E and $(d, D) \in \mathbb{R}^n \rtimes M_n(\mathbb{R}^n)$ be a suitable affine endomorphism. Then*

1. *D is either the zero $n \times n$ matrix 0_n or*
2. *D is an element of $\mathrm{Gl}(n, \mathbb{Q})$ such that in each row and each column of D there is exactly one non-zero element, which is an odd integer.*

Proof. To prove this proposition we distinguish three cases depending on the number of (a_i, A_i)'s which are mapped onto a pure translation:

- *Case 1*: Suppose that θ is a homomorphism of E such that two or more of the images of $(a_1, A_1), \ldots, (a_n, A_n)$ are pure translations. So there exist i and j, $i \neq j$, for which $\theta(a_i, A_i)$ and $\theta(a_j, A_j)$ are pure translations. Then Lemma 3.2 implies that $D = 0_n$.
- *Case 2*: Suppose that θ is a homomorphism of E such that just one of the images of $(a_1, A_1), \ldots, (a_n, A_n)$ is a pure translation. So part one of Lemma 3.2 implies that all the elements of D are even integers. But there also has to exist a j such that $\theta(a_j, A_j)$ is not a pure translation. So part two of Lemma 3.2 implies that there is a i such that d_{ij} is an odd integer. Since this gives a contradiction, we conclude that no such homomorphism exists.
- *Case 3*: Suppose that θ is a homomorphism of E such that none of the images of $(a_1, A_1), \ldots, (a_n, A_n)$ is a pure translation. In this situation, we can determine D completely. Namely, since $\theta(a_1, A_1)$ is not a pure translation, Lemma 3.2 implies that there exists a i_1 such that $d_{i_1 1}$ is odd, while the other elements of the i_1th row are zero. Doing the same for (a_2, A_2) we obtain a i_2 such that $d_{i_2 2}$ is odd, while the other elements of

the i_2th row are zero. Clearly $i_2 \neq i_1$, otherwise we would have that $d_{i_2 2}$ is zero on the one hand and an odd integer on the other hand. This can be done for all the images of $(a_1, A_1), \dots, (a_{n-1}, A_{n-1})$ and (a_n, A_n), so we have completely determined D as a matrix with exactly one non-zero, odd entry in each row and column. □

Remark 3.4. In the above proposition, we did not mention the image of the pure translations. But since the D always consist of integers, one can easily show that the image of a pure translation under the homomorphism again must be a pure translation.

Now that we have a clear view on the different possibilities for the linear parts of the suitable affine endomorphisms (d, D), we can start to use Theorem 2.3. Note that this theorem only uses the linear part D, so it is not a problem that we did not investigate the translational parts d appearing in a suitable affine endomorphism.

As a first step, in the following lemma we calculate the determinants which appear in Theorem 2.3 and in a second lemma we determine the signs of these determinants. To a $(n \times n)$-matrix D with in each row and each column exactly one non-zero element (as in the second part of Proposition 3.3), one can associate an unique permutation μ of n elements. Namely, for any $i = 1, 2, \dots, n$ let $\mu(i)$ be the unique index such that $d_{i\mu(i)} \neq 0$. Clearly μ is a permutation of n elements and μ has a unique cycle decomposition.

Lemma 3.5. *Suppose B is any diagonal matrix whose diagonal entries b_{ii} are 1's and −1's:*

1. *If $D = 0_n$ is the zero $(n \times n)$-matrix, then $\det(I_n - BD) = 1$ for all possible B.*
2. *Let D be a $(n \times n)$-matrix, such that in each row and each column of D there is exactly one non-zero element and let μ be the associated permutation. Let the cycle decomposition of μ be*

   $$(l_1^1 l_2^1 \cdots l_{p_1}^1)(l_1^2 l_2^2 \cdots l_{p_2}^2) \cdots (l_1^r l_2^r \cdots l_{p_r}^r).$$

   Then we have that

   $$\begin{aligned}\det(I_n - BD) &= \det(B) \times \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i \mu(l_1^i)} d_{l_2^i \mu(l_2^i)} \cdots d_{l_{p_i}^i \mu(l_{p_i}^i)}) \\ &= \det(B) \times \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i l_2^i} d_{l_2^i l_3^i} \cdots d_{l_{p_i}^i l_1^i}).\end{aligned}$$

Proof.

1. Trivial.
2. Since $B^2 = I_n$, we have that $\det(I_n - BD) = \det(B) \cdot \det(B - D)$. Now we show that

   $$\det(B - D) = \prod_{i=1}^{r} (b_{l_1^i l_1^i} b_{l_2^i l_2^i} \cdots b_{l_{p_i}^i l_{p_i}^i} - d_{l_1^i \mu(l_1^i)} d_{l_2^i \mu(l_2^i)} \cdots d_{l_{p_i}^i \mu(l_{p_i}^i)}).$$

   We do this by induction on n, the case $n = 1$ being trivial.

Suppose the formula holds for $(n-1)\times(n-1)$-matrices ($n\geq 2$), then we distinguish two cases:

(a) $d_{nn}\neq 0$ or equivalently $\mu(n)=n$. Then $\det(B-D)=(b_{nn}-d_{n\mu(n)})\det(B'-D')$, where B' (resp. D') is obtained from the matrix B (resp. D) by deleting the last row and column. By applying the induction hypothesis to $B'-D'$ we obtain the result.
(b) $d_{nn}=0$. Then there exists a unique k such that $d_{nk}\neq 0$ (or $\mu(n)=k$) and a unique l with $d_{ln}\neq 0$ (or $\mu(l)=n$). So in the cycle decomposition of μ, there is a cycle of the form $(l_1\cdots lnk\cdots l_p)$.

In the last column of the matrix $B-D$ there are two non-zero elements, namely b_{nn} and $-d_{ln}$.

Since $b_{nn}=\pm 1$ we can create a zero in the last column of the lth row by adding to the lth row $b_{nn}\cdot d_{ln}$ times the last row. In the computation below, this operation is used in the step indicated with $(*)$.

In this calculation, we again use B' and D' to denote the matrices obtained by deleting the last row and column of B and D, respectively. Note that the lth row of D' and the kth column of D' only consists of zeros and that in each other row and each other column of D' there is exactly one non-zero element.

Also, we need D'' obtained from D' by changing the kth component of the lth row of D' to $b_{nn}d_{ln}d_{nk}$. Since D' is from the above form, we then have that in each row and each column of D'' there is exactly one non-zero element. So

$$
\begin{aligned}
\det(B-D) &= \det\begin{pmatrix} & & & & & & & 0\\ & & & & & & & \vdots\\ & & & & & & & 0\\ & & & B'-D' & & & & -d_{ln}\\ & & & & & & & 0\\ & & & & & & & \vdots\\ & & & & & & & 0\\ 0 & \cdots & 0 & -d_{nk} & 0 & \cdots & 0 & b_{nn}\end{pmatrix}\\
&\overset{(*)}{=}\det\begin{pmatrix} & & & & & & & 0\\ & & & B'-D'' & & & & \vdots\\ & & & & & & & 0\\ 0 & \cdots & 0 & -d_{nk} & 0 & \cdots & 0 & b_{nn}\end{pmatrix}\\
&= b_{nn}\det(B'-D''). \qquad (2)
\end{aligned}
$$

Now we can associate a permutation μ'' of $n-1$ elements to D''. The cycle decomposition of μ'' is obtained from the cycle decomposition of μ by replacing in this decomposition the cycle $(l_1\cdots lnk\cdots l_p)$ which contains n, with the cycle $(l_1\cdots lk\cdots l_p)$. The induction hypothesis now applies to the matrix $B'-D''$ and it follows that all factors in the expansion of $\det(B'-D'')$ except the one containing the terms of the lth row are of the desired form. The exceptional factor containing the elements of the lth row and corresponding to the cycle

$(l_1 \cdots lk \cdots l_p)$ is of the following form:

$$b_{l_1l_1} \cdots b_{ll} \cdots b_{l_pl_p} - d_{l_1\mu(l_1)} \cdots (b_{nn}d_{ln}d_{nk}) \cdots d_{l_p\mu(l_p)}.$$

We can multiply the factor above with b_{nn} to get

$$b_{nn}b_{l_1l_1} \cdots b_{ll} \cdots b_{l_pl_p} - b_{nn}d_{l_1\mu(l_1)} \cdots (b_{nn}d_{ln}d_{nk}) \cdots d_{l_p\mu(l_p)},$$

which equals ($n = \mu(l)$ and $\mu(n) = k$)

$$b_{l_1l_1} \cdots b_{ll}b_{nn} \cdots b_{l_pl_p} - d_{l_1\mu(l_1)} \cdots d_{l\mu(l)}d_{n\mu(n)} \cdots d_{l_p\mu(l_p)}.$$

This part of the expansion of (2) together with the other factors show that $\det(B - D)$ is of the desired form. □

Note that the number of factors of the determinants in the previous lemma does not depend on B, but only on D (in fact, only on the form of D determined by μ). So for a given D, with D as in the second case of Lemma 3.2, and any diagonal matrix B consisting of 1's and -1's, we obtain that $\det(I_n - BD) = \det(B)(\pm 1 - x_1) \cdots (\pm 1 - x_k)$. Here $x_1, \ldots, x_k \in 1 + 2\mathbb{Z}$ and the ± 1's depend on the choice of B. In this perspective the following lemma is crucial.

Lemma 3.6. *Fix an integer $k \geq 1$ and $x_1, \ldots, x_k \in 1 + 2\mathbb{Z}$. Then*

$$\begin{array}{ll} \textit{either } (\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) \geq 0 & \textit{for all possible } \epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}, \\ \textit{or } (\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) \leq 0 & \textit{for all possible } \epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}. \end{array}$$

Proof. Suppose there exists $\epsilon_1, \ldots, \epsilon_k$ and $\epsilon'_1, \ldots, \epsilon'_k$ such that $(\epsilon_1 - x_1) \cdots (\epsilon_k - x_k) > 0$ and $(\epsilon'_1 - x_1) \cdots (\epsilon'_k - x_k) < 0$. This is only possible if there exist a j for which $1 - x_j > 0$ and $-1 - x_j < 0$ or conversely $1 - x_j < 0$ and $-1 - x_j > 0$. But in the former case we quickly see that $x_j = 0$ which is not possible and in the latter case we obtain the contradiction that $x_j > 1$ and $x_j < -1$. □

We are now ready to prove the Anosov theorem for flat orientable GHW manifolds.

Theorem 3.7. *Let $n \geq 3$ be an odd integer and M is an orientable n-dimensional flat generalized Hantzsche–Wendt manifold. Then for each continuous map $f : M \to M$ we have that $N(f) = |L(f)|$.*

Proof. Suppose $f : M \to M$ is a continuous map on M. Due to Corollary 2.2 we know that f is homotopic to a map g induced by an suitable affine endomorphism (d, D) of $\mathbb{R}^n$ and due to Proposition 3.3 we know how D looks like. Since the Nielsen numbers are homotopy invariants it suffices to prove the theorem for the map g. We use Theorem 2.3 to verify that $N(g) = |L(g)|$. Therefore we have to calculate $\det(I_n - T(x)D)$ for each $x \in F$. Note that for each $x \in F$, $T(x)$ is a diagonal matrix whose diagonal entries consist of an even numbers of -1's while the others are 1 and so $\det(T(x)) = 1$. Therefore we can apply Lemmas 3.5 and 3.6 to the determinants $\det(I_n - T(x)D)$ which finishes the proof of the theorem. □

4. Non-orientable flat GHW manifolds

We show for any non-orientable flat GHW manifold M that Theorem 3.7 does not hold. Again due to Theorem 2.6 we have that the manifold M is a diagonal manifold. So we can assume that $E = \pi_1(M)$ is generated by

$$(z_1, I_n), \dots, (z_n, I_n), (a_1, A_1), \dots, (a_{n-1}, A_{n-1})$$

with $z_i \in \mathbb{Z}^n$, A_i a diagonal matrix whose diagonal entries consist of 1's and -1's and the a_i are appropriate translational parts with 0 and 1/2's in their components [14, Lemma 1.4]. We can no longer be specific about the A_i's, since in the non-orientable case there are more possibilities. However as M is a non-orientable manifold, there exists a matrix A_j such that $\det(A_j) = -1$. We can prove the following theorem concerning non-orientable flat GHW manifolds.

Theorem 4.1. *If M is a non-orientable flat GHW manifold. Then there always exists a continuous map $f : M \to M$ for which $N(f) \neq |L(f)|$.*

Proof. Consider for example the affine transformation with translational part zero and

$$\text{linear part } D = \begin{pmatrix} 3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{pmatrix}.$$

To show that $(0, D)$ induces a continuous map $f : M \to M$, it suffices to verify that conjugation with $(0, D)$ maps E to E. Assume $(b, B) \in E$ (we know that $b \in (1/2)\mathbb{Z}^n$). If we conjugate with $(0, D)$ we obtain the following:

$$(0, D)(b, B)(0, D)^{-1} = (Db, DB)(0, D^{-1}) = (Db, DBD^{-1}).$$

Because D and B are diagonal matrices, $DBD^{-1} = B$ and since $Db = 3b$ we obtain that (b, B) is mapped onto $(3b, B) = (2b, I_n)(b, B) \in E$. Since $b \in (1/2)\mathbb{Z}^n$, we have that $(3b, B) \in E$ or conjugation with $(0, D)$ maps E to E.

To show that $N(f) \neq |L(f)|$, we again use Theorem 2.3. It is sufficient to find $x, x' \in F$ such that $\det(I_n - T(x)D) > 0$ and $\det(I_n - T(x')D) < 0$. We establish this using $\det(I_n - D)$ and $\det(I_n - A_jD)$ (with A_j, such that $\det A_j = -1$). One easily verifies that $\det(I_n - D) = (1-3)\cdots(1-3)$ and $\det(I_n - A_jD) = \det(A_j)\det(A_j - D) = (-1)(a_{11} - 3)\cdots(a_{nn} - 3)$ with $a_{ii} \in \{-1, 1\}$ for $1 \leq i \leq n$. So if we apply Lemma 3.6, we obtain that the two determinants have a different sign and therefore $N(f) \neq |L(f)|$. □

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